NONLINEAR TIME SERIES ANALYSIS, WITH APPLICATIONS TO MEDICINE

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LECTURE 2
DYNAMICAL SYSTEMS

OUTLINE

1. Generalities

Dynamical systems are models for phenomena evolving in time in a deterministic way.

Ingredients.

1. A *state space* Ω. A state $x \in \Omega$ is determined by d variables or "degrees of freedom".
2. *Time*, which may be
   - discrete: $n = 0, 1, ...$ (future), $n = -1, -2, ...$ (past)
   - continuous: $t \to \infty$ (future), $t \to -\infty$ (past)
3. *Evolution equation*:
   - discrete time: difference eqn. ($x_{n+1} = f(x_n)$).
   - continuous time: differential eqn. ($dx/dt = f(x(t))$).

We will mostly consider discrete-time dynamical systems.

1. Generalities

Notation for discrete-time DS: (Ω, f)

Historically f *was (and very often still is) supposed to be invertible because Newtonian mechanics is time reversible.*

A dynamical system can be

- *linear*, if the evolution equation is linear.
- *nonlinear*, otherwise.
- *conservative*, if the volume in state space is preserved in time
- *dissipative*, if the volume in state space is contracted in time.

1. Generalities

Examples of discrete-time systems

- The number of individuals of a population every day, year, generation, etc.
- Monthly interest earned by a saving deposit
- Average temperature in the last years, decades, centuries,...

Examples of continuous-time systems

- A particle moving in a force field
- Current or voltage in an electronic circuit
- Concentration of a compound during a chemical reaction

1. Generalities

Examples of state spaces.

1. Simple pendulum: $\Omega =$ circle
2. Double pendulum: $\Omega =$ circle $\times$ circle $=$ 2D torus
3. Mass point on the plane with polar coordinates: $\Omega = \mathbb{R}^3 \times$ circle $=$ cylinder
4. Mass point in space with Cartesian coordinates: $\Omega = \mathbb{R}^6$

1. Generalities

Forward evolution equation of a discrete-time system:
Let f a map from Ω to Ω: $f : \Omega \to \Omega$.

- Initial state (time $n = 0$): x_0
- Time $n = 1$: $x_1 = f(x_0)$.
- Time $n = 2$: $x_2 = f(x_1) = f(f(x_0)) \equiv f^2(x_0)$
- Time $n = k$: $x_k = f(x_{k-1}) = f^k(x_0)$, where

$$f^k(x) \equiv \underbrace{f(f(.....f(x)....))}_{k \text{ times}}$$

is called the *kth iterate* of f.

1. Generalities

The infinite sequence

$$x_0, x_1, ..., x_k, ... = x_0, f(x_0), ..., f^k(x_0), ... \equiv (f^n(x_0))_{n\geq 0},$$

is called the (*forward*) *orbit* of x_0. The state x_0 is the *initial condition*.

- If f is an *invertible* map, one can also define the *backward orbit*:

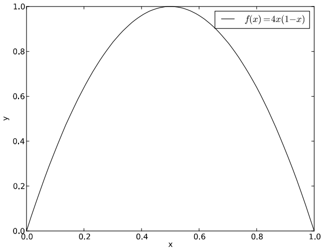
  $$x_0, x_{-1}, ..., x_{-k}, ... = x_0, f^{-1}(x_0), ..., f^{-k}(x_0), ... \equiv (f^{-n}(x_0))_{n\geq 0}.$$

  The *full orbit* is

  $$..., x_{-k}, ..., x_{-1}, x_0, x_1, ..., x_k, ... (f^n(x_0))_{n\in\mathbb{Z}}.$$

- *At the contrary that with random sequences, once x_0 is known its orbit is determined.*

Example. Take $\Omega = [0,1]$ and $f(x) = 4x(1-x)$, the *logistic map*.



$$\Rightarrow (f^n(0.6416))_{n\geq 0} = 0.6416,\ 0.9198,\ 0.2951,\ 0.8320, 0.5590,\ 0.9861, \ldots$$

Remark. For the logistic map

$$x_n = \tfrac{1}{2}\left[1 - \cos\left(2^n \cos^{-1}(1-2x_0)\right)\right],$$

but don't use this formula for computations!

Special orbits.

- A point x is a *fixed point*, *stationary point*, or *equilibrium point* if

$$f(x) = x.$$

- A point x is *periodic* of *period* p if the finite sequence

$$\{x, f(x), f^2(x), ..., f^{p-1}(x)\}$$

  repeats in its orbit. This means that $f^p(x) = x$.
- The continuous-time counterpart of a periodic cycle is called a *limit cycle*.

2. Attractors & dimensions

Fixed points and periodic cycles are examples of *invariant sets*.

A set $A \subset \Omega$ is called *invariant* if $f(A) \subset A$, i.e., no orbit starting in A can leave.

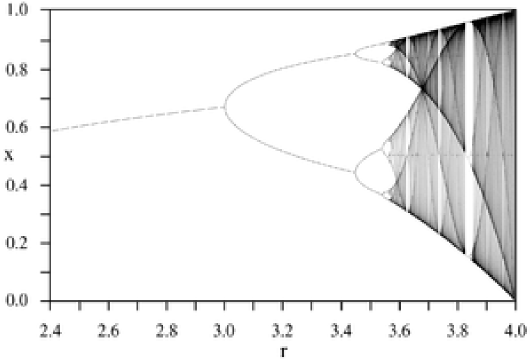
Definition. A *bounded* and *closed* set $A \subset \Omega$ is an *attractor* if

1. it is *absorbing* (i.e., absorbs all orbits starting in a neighborhood of A)
2. it is *invariant*

The greatest neighborhood absorbed by an attractor is called its *basin of attraction*.

Attractors are important because they contain the long-term dynamical behavior of the system.

Attractors of the logistic family, $f_r(x) = rx(1-x)$, $0 \leq x \leq 1$, $0 \leq r \leq 4$.

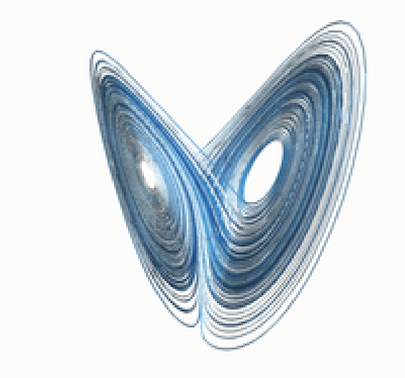


Henon attractor. *Henon map*:

$$H(x,y) = (1 - 1.4x^2 + 0.3y, x).$$

Lorenz attractor. *Lorenz system*:

$$(dx/dt, dy/dt, dz/dt) = (-10(x-y), 28x - y - xz, -\tfrac{8}{3}z + xy)$$



2. Attractors & dimensions

Main characterizations of an attractor:

- *Dimensions* (or active degrees of freedom)
  - Box-counting dimension
  - Information dimension
  - Generalized dimensions (correlation dimension, etc.)
- *Lyapunov exponents*

The box counting dimension is a poor man's version of the conventional geometric dimension (*Hausdorff dimension*).

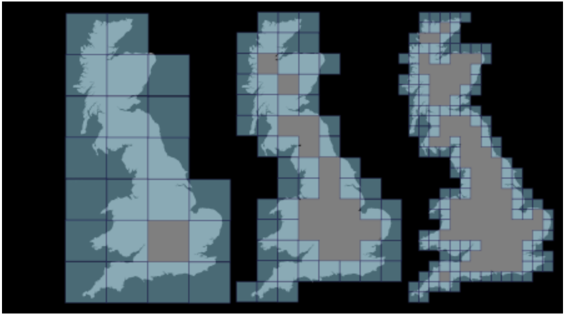
Method. Put a grid of size s on the structure and count the number $N(s)$ of grid boxes which contain points of the structure. Its *box-counting dimension* is

$$D_0 = \lim_{s\to 0} \frac{N(s)}{\log(1/s)}.$$

- If you plot $N(s)$ vs $\log(1/s)$, then D_0 is the slope of the resulting straight line.
- In practice, the orbits have a finite length n.

$$N(s,n) \approx N(s) - const \cdot s^{-\alpha} n^{-\beta}.$$

Example. For coast of the UK, $D_0 = 1.31$.



2. Attractors & dimensions

The information dimension takes into account the number of points inside the grid boxes.

Definition. Let $B_1, ..., B_{N(s)}$ the boxes of a grid of size s containing points of an attractor A, and $\mu(B_j)$ the relative count of points in B_j. The *information needed to locate a point in the attractor with precision* s is

$$I(s) = -\sum_{j=1}^{N(s)} \mu(B_j) \log \mu(B_j).$$

The *information dimension* of A, D_1, is

$$D_1 = \lim_{s \to 0} \frac{I(s)}{\log(1/s)}$$

D_0 and D_1 are just the two first instances of a whole hierarchy of dimensions. Set

$$I_q(s) = \frac{1}{1-q} \log \sum_{j=1}^{N(s)} \mu(B_j)^q,$$

where $q \geq 0$. The *Renyi dimension* of the attractor is

$$D_q = \lim_{s \to 0} \frac{I_q(s)}{\log(1/s)}.$$

D_2 is called the *correlation dimension*.

2. Attractors & dimensions

Property. $D_0 \geq D_1 \geq ... \geq D_q \geq D_{q+1} \geq ...$

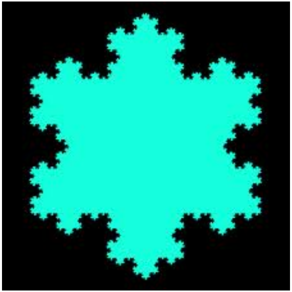
Example. For the Henon attractor:

$$D_0 = 1.28 \pm 0.01;\ D_1 = 1.23 \pm 0.02;\ D_2 = 1.21 \pm 0.01.$$

Attractors with fractional dimensions are called *strange attractors*. They are typical of chaotic dissipative maps and flows.

Fractional dimensions are typical of self-similar objects (*fractal geometry*).

Example. The *Koch snowflake* (fractal dimension $= \log 4/\log 3 \simeq 1.26$)



3. Lyapunov exponents

Definition. The map f has *sensitive dependence on initial conditions* if for any $x_0 \in \Omega$ there is other $y_0 \in \Omega$ arbitrarily closed such their orbits diverge from each other at some time (they can join later).

This property is quantified by the *maximal Lyapunov exponent* λ:

$$\left. \begin{array}{l} \text{dist}\,(x_0, y_0) = \delta_0 \ll 1 \\ \text{dist}\,(x_n, y_n) = \delta_n \end{array} \right\} \Rightarrow \delta_n \simeq \delta_0 e^{\lambda n}$$

Thus $\lambda > 0$ amounts to an exponential divergence of nearby orbits.

Remarks.

- In general, $\lambda = \lambda(x_0)$
- There are so many Lyapunov exponents as directions in Ω $(\dim \Omega)$.

Calculation of λ.

- If Ω is a 1D interval and f is differentiable,

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| f'(x_k) \right|,$$

  where $x_k = f^k(x_0)$.

- In general, one has to use numerical algorithms (quite tricky!).

3. Lyapunov exponents

Definition. An attractor A is called *chaotic* if f has sensitive dependence on initial conditions taken on A.

Relation between λ and the attractor dynamics.

stable fixed point	$\lambda < 0$
stable limit cycle	$\lambda = 0$
chaos	$0 < \lambda < \infty$
noise	$\lambda = \infty$

A measure generalizes the concept of length, area, and volume.

- **Lebesgue measure**. Let $B \subset \mathbb{R}^d$:

$$\textit{Lebesgue measure} \text{ of } B \equiv \int_B dx \quad [\text{Shorthand: } \mu(dx) = dx]$$

$=$ length $(d = 1)$, area $(d = 2)$, volume $(d = 3)$ of B.

- **Stieltjes measure**. Let ρ be a *density* (i.e. $\rho(x) \geq 0$, $\int_\Omega \rho(x) = 1$):

$$\textit{Stieltjes measure} \text{ of } B \equiv \int_B \rho(x)dx \quad [\text{Shorthand: } \mu(dx) = \rho(x)dx]$$

Used to calculate the mass, charge, etc. of continuous distributions.

- **Dirac measure**. Let $\omega \in \Omega$ (the "support"):

$$\delta_\omega(B) = \begin{cases} 1 & \text{if } \omega \in B, \\ 0 & \text{if } \omega \notin B. \end{cases} \Rightarrow \int_\Omega f(x)d\delta_\omega(x) = f(\omega).$$

Used to calculate the mass, charge, etc. of discrete distributions.

4. Invariant measures

In dynamical systems a measure μ *must have two properties:*

1. **Normalization**, i.e., $\mu(\Omega) = 1$ (hence, it is formally a probability).
2. **Invariance**, i.e.,
$$\mu(f^{-1}(B)) = \mu(B),$$
where $B \subset \Omega$ and $f^{-1}(B)$ is the set of all predecessors of points in B.

A dynamical system (Ω, f) endowed with a normalized and invariant measure μ is called a *measure-preserving system*, and denoted by

$$(\Omega, f, \mu).$$

Measure-preserving systems are the deterministic counterpart of stationary random systems.

4. Invariant measures

There are plenty of invariant measures.

- If x_0 is a fixed point,

$$\mu = \delta_{x_0}$$

- If $x_0, x_1, ..., x_N$ is a periodic cycle,

$$\mu = \frac{1}{N+1} \sum_{k=0}^{N} \delta_{x_k}$$

But they are physically unobservable in general, and the dynamic is uninteresting anyway.

4. Invariant measures

When defining D_1, we considered the quantities

$$\mu(B) = \lim_{n\to\infty} \frac{\#\{x_0, x_1..., x_{n-1} \in B\}}{n}, \tag{1}$$

where $x_1 = f(x_0)$, $x_2 = f^2(x_0)$, etc.

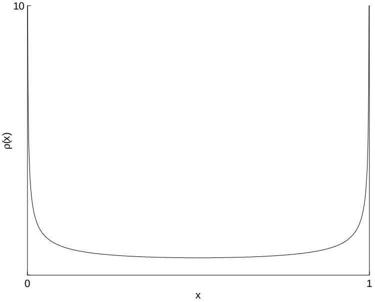
Fact. On chaotic attractors, (1) defines a normalized and invariant measure.

This measure is called the *natural, physical, or empirical measure*.

The natural measure is the invariant measure used in applications.

Example. The natural measure of the *logistic map* is

$$\mu(B) = \int_B \frac{dx}{\pi\sqrt{x(1-x)}}, \text{ i.e., } \rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$



$\rho(x)$ *can be visualized as the histogram of a generic orbit.*

4. Invariant measures

If an invariant measure cannot be decomposed into further invariant pieces, it is called *ergodic*.

Ergodic Theorem (Birkhoff)**.** If the invariant measure μ is ergodic, then for every continuous map $\varphi : \Omega \to \mathbb{R}$,

$$\underbrace{\int_{\Omega} \varphi(x) d\mu(x)}_{\text{space average of } \varphi} = \underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x_0))}_{\text{time average of } \varphi}$$

for almost all x_0 (i.e., the exceptions build a set of μ-measure 0).

Properties that hold except for a set of points with μ-measure 0, all called *generic*.
E.g., the value of λ is generic for chaotic attractors.

Dynamical systems with chaotic attractors.

(**1D**-1) The *logistic* (or quadratic) *family*: $x_{n+1} = f_r(x_n)$, where

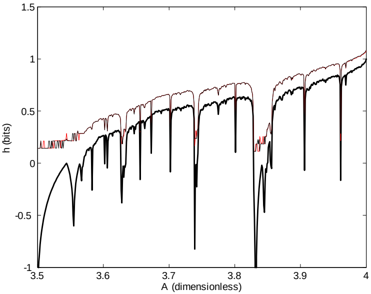
$$f_r(x) = rx(1-x),\ 0 \leq x \leq 1,\ 0 \leq r \leq 4.$$

- $f_4(x) = 4x(1-x)$ is the *logistic map*.
- The attractor of f_4 is $A = [0,1]$.
- Lyapunov exponent of $f_4(x)$:

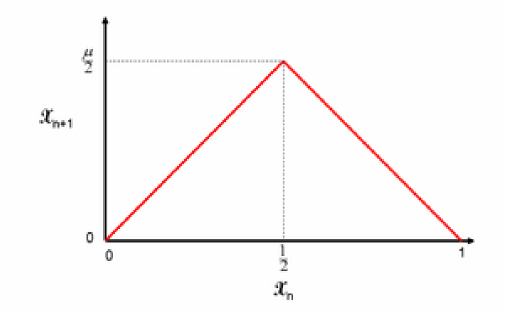
$$\lambda = \int_0^1 \left| f'(x) \right| d\mu(x) = \int_0^1 \frac{\log|4(1-2x)|}{\pi\sqrt{x(1-x)}} dx = \log 2.$$

5. Chaotic attractors

- f_r can be chaotic only if $r \geq r_\infty := 3.5699456718...$, the *Feigenbaum point*.
- For $r_\infty < r \leq 4$ there are infinite "periodic windows".
- Lyapunov exponents of the logistic family (*thick line*).

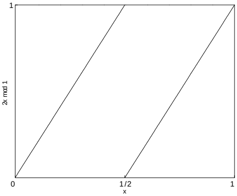


(**1D**-2) The *tent map* (1-dimensional “baker's map”)



$\lambda = \log 2$, $\mu =$ Lebesgue measure

(**1D**-3) The *binary shift map*:



$$\lambda = \log 2,\ \mu = \text{ Lebesgue measure}$$

Remark. The logistic, tent, and binary shift maps are 'isomorphic' (i.e., equivalent) in the sense of dynamical systems[1]. Thus:

- Orbits and invariant measures go into each other by a change of coordinates.
- They have the same Lyapunov exponents.

[1] J.M.A, *Permutation Complexity in Dynamical Systems*, Springer Verlag, 2010.

5. Chaotic attractors

(**D2**-1) *Henon map*:

$$(x_{n+1}, y_{n+1}) = (1 - 1.4x_n^2 + 0.3y_n,\ x_n)$$

$$\lambda = 0.419222 \pm 0.000003.$$

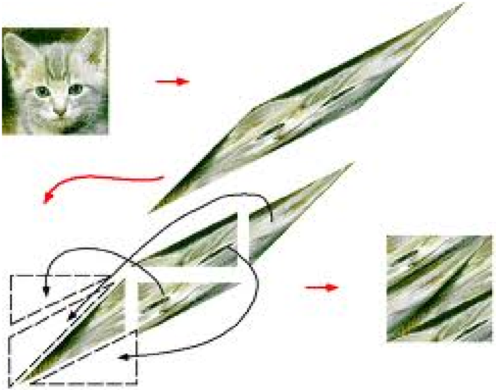
The Henon map is dissipative, with an area contraction factor of 0.3.

(**D2**-2) *Baker's map*: For $0 \leq x_n y_n \leq 1$,

$$(x_{n+1}, y_{n+1}) = \begin{cases} (2x_n,\ y_n) & \text{if } 0 \leq x_n < \frac{1}{2} \\ (2x_n - 1,\ \frac{1}{2}(y_n + 1)) & \text{if } \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

$$\lambda = \log 2,\ \mu = \text{ Lebesgue measure}$$

(**D2**-2) *Cat map*: $x_{n+1} = 2x_n + y_n$ (mod1), $y_{n+1} = x_n + y_n$, (mod1).

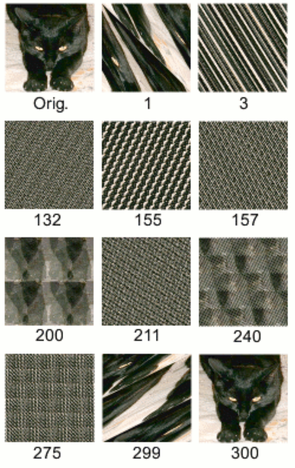


$$\lambda = \ln \frac{3+\sqrt{5}}{2} = 0.96242365...$$

The cat map is conservative.

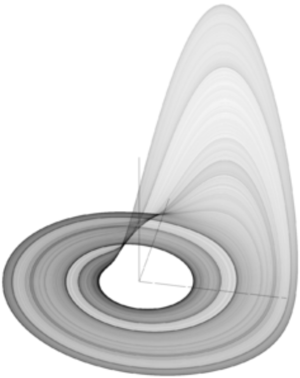
5. Chaotic attractors

Question: Is the cat map really chaotic?



5. Chaotic attractors

(**3D**-1) *Lorenz flow*. It is highly dissipative.
(**3D**-2) *Rössler flow*. $dx/dt = -y - x$, $dy/dt = x + ay$, $dz/dt = b + z(x - c)$.



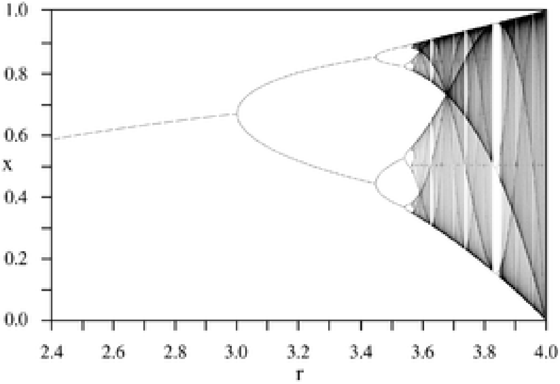
Some nonlinear phenomena:

1. *Coexistence of attractors* (multistability).
   - Description: f has several attractors.
   - Problem for TSA: Repetition of the experiment with the same parameters may yield a qualitatively different result.
2. *Intermittency*.
   - Description: Orbits alternate between periodic (regular, laminar) and chaotic (irregular, turbulent) behavior.
   - Problem for TSA: Different time scales.
3. *Bifurcations* (or phase transitions).
   - Description: Abrupt change of the attractor geometry at a critical value of a control parameter.
   - Problem for TSA: Change of stability.

A few typical bifurcations.

- *Pitchfork bifurcation*. A stable fixed point becomes unstable and two new stable fixed points are created.
- *Tangent* (or saddle-node) *bifurcation*. It consists in the creation of two periodic orbits, one stable and one unstable. Tangent bifurcation is the mechanism for one type of intermittency (type I).
- *Supercritical Hopf bifurcation*. At the parameter value where a stable fixed point becomes unstable, a stable limit cycle is born.

Example: Feigenbaum bifurcations.



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3. R.C. Robinson, *An introduction to dynamical systems*, Prentice Hall, 2004.
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